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Nonholonomic left and right flows on Lie groups

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Abstract. Nonholonomic systems on Lie groups with a left-invariant Lagrangian and a right-invariant nonintegrable distribution are studied. A method for construction of the first integrals is given. It is applied to three-dimensional Lie groups and to the four-dimensional nonholonomic rigid body motion. New integrable problems with an invariant measure are obtained.

0. Introduction

Some of the most interesting remarks about nonholonomic systems were given long ago by Chaplygin. He noticed that some systems, although not Hamiltonian, had an invariant measure. Then, if the phase space is foliated with the two-dimensional invariant manifolds, the equations of motion can be integrated by quadratures according to the Jacobi theorem. Some of those solvable systems are very close to the integrable Hamiltonian systems: the motion is unevenly winding over the two-dimensional tori. The review of integrable examples can be found in [1, 11].

In this paper, we are going to consider nonholonomic systems on Lie groups with a left-invariant Lagrangian and a right-invariant nonholonomic distribution (LR systems) [7, 15]. In section 1 we shall set the notation. A method for the construction of the integrals of an LR system by using the invariants of the co-adjoint action is given in section 2. Both homogeneous and nonhomogeneous constraints are considered. These results are used for the construction of new integrable LR systems with an invariant measure on the three-dimensional groups in section 3. In section 4, we consider the LR systems on $SO(4)$. Explicit formulae are given for the conservation laws of the four-dimensional nonholonomic rigid body motion.

Let us note that nonholonomic systems on Lie groups with left-invariant constraints also appear in natural mechanical problems [7–9]. The equations of motion of a mechanical nonholonomic system are not equivalent to the equations obtained from a nonholonomic variational problem. The variational problems on Lie groups have been studied in [14].

1. Nonholonomic LR systems

The LR systems have been introduced in [15]. A descriptive and motivating example is the Veselov–Veselova rigid body problem. Another important example is the Chaplygin problem of the rolling of a balanced, dynamically asymmetric ball on a rough surface [4].

Example 1.1. *The Veselov–Veselova problem represents the rotation of a rigid body fixed at a point with the constraint* $\langle \vec{N}, \vec{\Omega} \rangle = 0$ *where* $\vec{\Omega}$ *is the angular velocity and* $\vec{N}$ *is a constant* *vector in the fixed reference frame [15]. Let I be the inertia operator. The equations of motion under inertia in body coordinates are*

$$\begin{aligned} &\dot{\vec{M}} = \vec{M} \times \vec{\Omega} + \lambda \vec{N} \qquad \dot{\vec{N}} = \vec{N} \times \vec{\Omega} \qquad \langle \vec{N}, \vec{\Omega} \rangle = 0 \\ &\lambda = \langle \vec{N}, A\vec{N} \rangle^{-1} \langle \vec{M} \times A\vec{M}, A\vec{N} \rangle \qquad \vec{M} = I\vec{\Omega} \qquad A = I^{-1}. \end{aligned} \tag{1}$$

The motion is unevenly winding over the two-dimensional invariant tori. Also, the integrability remains even after addition of a symmetric gyroscope (then $\vec{M} = I\vec{\Omega} + \vec{P}$) [15]. Fedorov extended the integration of the equations (1) to the case of the nonhomogeneous constraint $\langle \vec{N}, \vec{\Omega} \rangle = q$ [6].

Definition 1.1. *The LR system is the nonholonomic Lagrangian system (G, L, D) where G is the n-dimensional Lie group, L is the left-invariant Lagrangian and $D \subset TG$ is the right-invariant $(n-k)$-dimensional nonintegrable distribution.*

The distribution can be given by the k independent right-invariant one-forms $\alpha^i \in \Lambda^1(G)$ in the following way:

$$D_g = \{\xi \in T_g G,\ (\alpha^i_g, \xi) = 0,\ (R_g)^* \alpha^i_g = n^i = \text{const},\ i = 1, \ldots, k < n\}$$

where $(\,,\,)$ denotes the pairing between the $T^*_g G$ and $T_g G$, and R_g denotes the right multiplication with g, $R_g(a) = ag$, $a \in G$. The distribution is nonintegrable if and only if $[\mathcal{D}, \mathcal{D}] \not\subset \mathcal{D}$, where $\mathcal{D} = D_e$.

The admissible paths satisfy, in general, nonhomogeneous right-invariant constraints $(\alpha^i_g, \dot{g}) = q^i = \text{const}$. If $g(t)$ is a smooth path in G, we introduce as usual $\Omega(t) = (L_{g^{-1}})_* \dot{g}(t)$ a smooth path in Lie algebra $\mathcal{G} = T_e G$ and $N^i(t) = (L_g)^* \alpha^i_{g(t)} = Ad^*_{g^{-1}} n^i \in \mathcal{G}^*$. In this notation the constraints become

$$(\alpha^i_g, \dot{g}) = (\alpha^i_g, (L_g)_* \Omega) = (n^i, Ad_g \Omega) = (N^i, \Omega) = q^i \qquad i = 1, \ldots, k.$$

The equations of motion are derived from *the d'Alembert–Lagrange principle*

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{g}} - \frac{\partial L}{\partial g} = \sum \lambda_i \alpha^i_g \qquad (\alpha^i_g, \dot{g}) = q^i \quad i = 1, \ldots, k \tag{2}$$

where the Lagrange multipliers λ_i are chosen such that $(\alpha^i_g, \dot{g}) = q^i = \text{const}$.

We shall take the Lagrangian as the sum of the kinetic energy and the gyroscopic term. It can be written in the form $L = \frac{1}{2}(I\Omega, \Omega) + (P, \Omega)$, where $P \in \mathcal{G}^*$ and $I : \mathcal{G} \to \mathcal{G}^*$ is the symmetric positive definite operator which defines the left-invariant metric. Let $A = I^{-1} : \mathcal{G}^* \to \mathcal{G}$ and $M = \partial L / \partial \Omega = I\Omega + P$.

The equations of motion are reduced to

$$\begin{aligned} &\dot{M} = ad^*_\Omega M + \sum \lambda_i N^i \\ &\dot{N}^i = ad^*_\Omega N^i \\ &(N^i, \Omega) = q^i \qquad i = 1, \ldots, k \qquad \Omega = A(M - P). \end{aligned} \tag{2$'$}$$

In particular, for $G = SO(3)$ equations (2′) take the form (1).

Recall that a Lie group G is called *unimodular* if it has a two-side invariant measure. The criterion for a group to be unimodular is that the structural constants of its Lie algebra satisfy the equations $\sum_k C^k_{ik} = 0$ for all i. The Lie algebra is also called *unimodular*.

The important characteristic of LR systems on unimodular groups is the existence of an invariant measure. Then, equations (2′) have the invariant measure $\mu(N)\mathrm{d}M\, \mathrm{d}N^1 \ldots \mathrm{d}N^k$, where $\mu(N) = \sqrt{\det(N^i, AN^j)}$ [16].

2. The first integrals

In this section we shall consider the LR systems with one constraint:

$$(\alpha_g, \dot{g}) = (N, \Omega) = q \qquad [\mathcal{D}, \mathcal{D}] = \mathcal{G} \qquad \mathcal{D} = \{\Theta \in \mathcal{G},\ (n, \Theta) = 0\}$$

in the presence of an additional force with potential $V = V(g)$. Then the Lagrangian is $L = \frac{1}{2}(I\Omega, \Omega) + (P, \Omega) - V(g)$. Suppose that V is a G_n-invariant function, where $n = (R_g)^*\alpha = \text{const}$ and $G_n = \{g \in G,\ Ad^*_{g^{-1}} n = n\}$. Then $V(g)$ induces a well-defined function $U(N)$ on the orbit of the co-adjoint action $\mathcal{O}(n) \subset \mathcal{G}^*$: $U(N) = V(g)$, $N = Ad^*_{g^{-1}} n \in \mathcal{O}(n)$. An example is the gravity potential in a rigid body motion.

If we consider n as variable parameter, $U(N)$ becomes the function on $\mathcal{G}^*$. We shall use such functions as potentials. In the space $\mathcal{M} = \mathcal{G}^*\{M\} \times \mathcal{G}^*\{N\}$ the reduced equations of motion take the following form [16]:

$$\begin{aligned} &\dot{M} = ad^*_\Omega M + ad^*_{dU} N + \lambda N \\ &\dot{N} = ad^*_\Omega N \\ &(N, \Omega) = q \qquad \Omega = A(M - P) \\ &\lambda = -(N, AN)^{-1}(ad^*_\Omega M + ad^*_{dU} N, AN). \end{aligned} \tag{3}$$

The adding of the potential has no influence to the existence of an invariant measure. Detailed derivations of equations (3) without constraint are given in [13]. These are the Euler–Poincaré equations for the semidirect product of the group G and its Lie algebra $\mathcal{G}$.

We are interested in the construction of the integrals of equations (3). Recall that a function $J : \mathcal{G}^* \to R$ is an invariant of co-adjoint action if it satisfies

$$(M, [\mathrm{d}J(M), \Theta]) = 0 \qquad \text{for all} \quad M \in \mathcal{G}^*, \quad \Theta \in \mathcal{G}. \tag{4}$$

Lemma 2.1. *If $J : \mathcal{G}^* \to R$ is an invariant on $\mathcal{G}^*$, then $J_1(M, N) = J(N)$ is an integral of equations (3). For the Hamiltonian $H(M, N) = \frac{1}{2}(M - P, \Omega) + U(N)$, we have $\dot{H} = \lambda(N, \Omega) = \lambda q$. It is preserved only when the constraint is homogeneous: $q = 0$.*

Lemma 2.2. *(i) Let $J : \mathcal{G}^* \to R$ be an invariant on $\mathcal{G}^*$ of the form $J(M) = \Phi(M, M)$, where Φ is a symmetric 2-tensor. Then along the trajectories of the system (3) the following holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(M, N) = \lambda\Phi(N, N).$$

(ii) If J is of the form $J(M) = (M, \eta)$, $\eta \in \mathcal{G}$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}(M, \eta) = \lambda(N, \eta).$$

Proof. Let $(e_1, \ldots, e_n)$ be a base of the algebra $\mathcal{G}$ with structural constants C^k_{ij}: $[e_i, e_j] = C^k_{ij} e_k$. Introducing the coordinates $M_k = (M, e_k)$, $N_k = (N, e_k)$, $\Omega = \Omega^k e_k$, equations (3) become

$$\begin{aligned} &\dot{M}_k = C^i_{jk} M_i \Omega^j + C^i_{jk} N_i \frac{\partial U}{\partial N_j} + \lambda N_k \\ &\dot{N}_k = C^i_{jk} N_i \Omega^j \\ &N_i \Omega^i = q \qquad \Omega^i = A^{ij}(M_j - P_j). \end{aligned} \tag{3$'$}$$

The derivative of $\Phi(N, M) = \Phi^{ij} N_i M_j$ along the trajectory is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(N, M) &= C^i_{jk} M_i \Omega^j \Phi^{kg} N_g + C^i_{jk} N_i \frac{\partial U}{\partial N_j} \Phi^{kg} N_g + \lambda N_k \Phi^{kg} N_g + C^i_{jk} N_i \Omega^j \Phi^{kg} M_g \\ &= \Omega^j (C^i_{jk} \Phi^{kg} \{M_i N_g + N_i M_g\}) + \frac{\partial U}{\partial N_j} (C^i_{jk} \Phi^{kg} N_i N_g) + \lambda J(N). \end{aligned} \tag{5}$$

Since $J(M)$ is an invariant, equation (4) gives

$$\sum C^i_{jk} M_i \frac{\partial J}{\partial M_k} = 0 \qquad j = 1, \dots n. \tag{4'}$$

From (4′) we get

$$\sum C^i_{jk} \Phi^{kg} M_i M_g = 0 \qquad j = 1, \dots n. \tag{6}$$

The tensor $M_i M_g$ is symmetric in i, g. Hence from (6) the sum $\sum C^i_{jk}\Phi^{kg}$ has to be anti-symmetric in i, g for all $j = 1, \dots n$. Then the first and the second sums in (5) are equal to zero as products of a symmetric and an anti-symmetric tensor in indices i, g.

In the second case, when $J(M) = (M, \eta) = M_k \eta^k$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(M, \eta) = C^i_{jk}\eta^k \left\{ M_i \Omega^j + N_i \frac{\partial U}{\partial N_j} \right\} + \lambda N_k \eta^k.$$

The equations (4′) give $C^i_{jk}\eta^k = 0$ for all i, j. This proves the second part of the lemma. □

From lemmas 2.1 and 2.2 we can get the following theorem.

Theorem 2.1. *(i) Let $J : \mathcal{G}^* \to R$ be an invariant on $\mathcal{G}^*$ of the form $J = \Phi(M, M)$, where Φ is a symmetric 2-tensor. Then equations (3) have the first integrals*

$$J_1 = \Phi(N, N)$$
$$J_2 = \Phi(N, N)(\tfrac{1}{2}(M - P, \Omega) + U(N)) - \Phi(M, N)(N, \Omega).$$

In the case when $U(N) = 0$ we have the four-degree polynomial integral

$$J_3 = \Phi(M, M)\Phi(N, N) - \Phi(M, N)^2.$$

(ii) If J is of the form $J = (M, \eta)$, $\eta \in \mathcal{G}$ then the integrals are

$$J_1 = (N, \eta)$$
$$J_2 = (N, \eta)(\tfrac{1}{2}(M - P, \Omega) + U(N)) - (M, \eta)(N, \Omega).$$

Remark 2.1. The conservation laws of the nonreduced system (2) can be obtained from the nonholonomic Noether theorem [1, 3, 7].

Suppose that V is a K-invariant function, where $K \subset G$ acts by left multiplication on G. The infinitesimal generator of the action is $v_Y(g) = (R_g)_* Y$, where $Y \in \mathcal{K} = T_e K$. The momentum of the system relative to the Lie group K is, by definition, the mapping $\mathcal{I}$: $TG \to \mathcal{K}^*$:

$$\mathcal{I}(\dot g, g | Y) = \left(\frac{\partial L}{\partial \dot g}, v_Y(g)\right) = \left((R_g)^* \frac{\partial L}{\partial \dot g}, Y\right) = (Ad^*_g M, Y) \qquad M = I\Omega + P.$$

Let $K' \subset K$ be the maximal subgroup whose action is consistent with the constraints: $v_Y(g) \in D_g$, $g \in G$, $Y \in \mathcal{K}' = T_e K'$. Then, we have $(\alpha(g), v_Y(g)) = (\alpha(g), (R_g)_* Y) = (n, Y) = 0$, which implies that $\mathcal{K}' = \mathcal{K} \cap \mathcal{D}$. From the Noether theorem follows that $\mathcal{J}(\dot g, g) = (Ad^*_g M)|_{\mathcal{K}\cap\mathcal{D}}$ is the integral of the system (2).

3. LR systems on three-dimensional groups

Let us recall the well known classification of the three-dimensional unimodular Lie algebras. Let (e_1, e_2, e_3) be a base of such algebra. Then, up to an isomorphism, there are only the following unimodular algebras (see, for example, [14]):

(i) the algebra of the 3×3 skew-symmetric matrices $so(3, R)$: $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$;
(ii) the algebra of the 2×2 matrices with zero trace $sl(2, R)$: $[e_1, e_2] = -e_3$, $[e_1, e_3] = 2e_1$, $[e_2, e_3] = -2e_2$;
(iii) the Heisenberg algebra h: $[e_1, e_2] = e_3$, $[e_1, e_3] = [e_2, e_3] = 0$;
(iv) the commutative algebra $t(3)$: $[e_i, e_j] = 0$.

There are no nonintegrable right(left)-invariant distributions on the groups corresponding to the commutative algebra $t(3)$. So, the nonholonomic LR systems with an invariant measure appear on three-dimensional groups G in the cases (i), (ii) or (iii).

In [16] it was proved that in the six-dimensional phase space T^*G the motion under inertia with a homogeneous constraint acts over the two-dimensional invariant surfaces. It can be solved by quadratures using the Jacobi theorem.

We shall consider the case of a nonhomogeneous constraint in the additional potential force field (equations (3)). In the six-dimensional space $\mathcal{M}\{M, N\}$ we have an invariant measure and the constraint $(N, \Omega) = q$. Thus we need three integrals of motion for the complete integrability of the system (3).

(i) $\mathcal{G} = so(3, R)$. Let $U(N) = 0$. From theorem 2.1 we get the well known integrals of the equations (1) [6, 15]:

$$\begin{aligned} J_1 &= \langle \vec{N}, \vec{N} \rangle \\ J_2 &= \tfrac{1}{2}\langle \vec{N}, \vec{N} \rangle \langle \vec{M} - \vec{P}, \vec{\Omega} \rangle - \langle \vec{N}, \vec{M} \rangle \langle \vec{N}, \vec{\Omega} \rangle \\ J_3 &= \langle \vec{N}, \vec{N} \rangle \langle \vec{M}, \vec{M} \rangle - \langle \vec{M}, \vec{N} \rangle^2. \end{aligned} \tag{7}$$

The integrable potential perturbations of the equations (1) were given in [5, 15].

(ii) $\mathcal{G} = sl(2, R)$. In the base (ii) the motion has the following form:

$$\begin{aligned} \dot{M}_1 &= M_3\Omega^2 - 2M_1\Omega^3 + N_3\frac{\partial U}{\partial N_2} - 2N_1\frac{\partial U}{\partial N_3} + \lambda N_1 \\ \dot{M}_2 &= 2M_2\Omega^3 - M_3\Omega^1 + 2N_2\frac{\partial U}{\partial N_3} - N_3\frac{\partial U}{\partial N_1} + \lambda N_2 \\ \dot{M}_3 &= 2M_1\Omega^1 - 2M_2\Omega^2 + 2N_1\frac{\partial U}{\partial N_1} - 2N_2\frac{\partial U}{\partial N_2} + \lambda N_3 \\ \dot{N}_1 &= N_3\Omega^2 - 2N_1\Omega^3 \\ \dot{N}_2 &= 2N_2\Omega^3 - N_3\Omega^1 \\ \dot{N}_3 &= 2N_1\Omega^1 - 2N_2\Omega^2 \\ N_1\Omega^1 &+ N_2\Omega^2 + N_3\Omega^3 = q \qquad \Omega^i = A^{ij}(M_j - P_j). \end{aligned} \tag{8}$$

An invariant of the co-adjoint action is $J(M) = 4M_1M_2 + M_3^2$. We always have two integrals of equations (8):

$$\begin{aligned} J_1 &= 4N_1N_2 + N_3^2 \\ J_2 &= J_1(N)(\tfrac{1}{2}(M - P, \Omega) + U(N)) - (N, \Omega)(2M_2N_1 + 2N_2M_1 + M_3N_3). \end{aligned} \tag{9}$$

In the case $U(N) = 0$, the problem is integrable. From theorem 2.1 we have the third integral

$$J_3 = (4N_1N_2 + N_3^2)(4M_1M_2 + M_3^2) - (2M_2N_1 + 2N_2M_1 + M_3N_3)^2. \tag{10}$$

The integrable potential perturbations can be found by the method from [5]. Looking for the potentials $U(N)$ for which there exists an integral of the form $\tilde{J}_3 = J_3 + V(N)$ we can get the following theorem.

Theorem 3.1. *Equations (8) are integrable for the potential*

$$\frac{\epsilon}{2}\left(4\frac{N_1^2}{a_2} + 4\frac{N_2^2}{a_1} + \frac{N_3^2}{a_3}\right)$$

where $A = \operatorname{diag}(a_1, a_2, a_3)$ *and* $P = 0$. *The third integral is*

$$\tilde{J}_3 = J_3 + 4\epsilon(4N_1N_2 + N_3^2)\left(\frac{N_1^2}{a_2a_3} + \frac{N_2^2}{a_1a_3} + \frac{N_3^2}{a_1a_2}\right)$$

where J_3 *is given by (10).*

Remark 3.1. The function $\frac{\epsilon}{2}(4\frac{N_1^2}{a_2} + 4\frac{N_2^2}{a_1} + \frac{N_3^2}{a_3})$ is the analogue of the Klebsch–Tisserand potential in a rigid body motion. With such a potential problem (8) is integrable even without the nonholonomic constraint.

(iii) $\mathcal{G} = h$. In the base (iii) equations (3) and integrals J_1 i J_2 are:

$$\begin{aligned}
&\dot{M}_1 = -M_3\Omega^2 - N_3\frac{\partial U}{\partial N_2} + \lambda N_1\\
&\dot{M}_2 = M_3\Omega^1 + N_3\frac{\partial U}{\partial N_1} + \lambda N_2\\
&\dot{M}_3 = \lambda N_3\\
&\dot{N}_1 = -N_3\Omega^2 \qquad \dot{N}_2 = N_3\Omega^1 \qquad \dot{N}_3 = 0\\
&N_1\Omega^1 + N_2\Omega^2 + N_3\Omega^3 = q \qquad \Omega^i = A^{ij}(M_j - P_j)
\end{aligned} \tag{11}$$

$$\begin{aligned}
&J_1 = N_3\\
&J_2 = N_3(\tfrac{1}{2}(M - P, \Omega)) + U(N)) - M_3(N, \Omega).
\end{aligned} \tag{12}$$

Let $U(N) = 0$. Contrary to the previous examples, theorem 2.1 does not give the third integral. However, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(N, AN) = 2N_3(\Omega^1 A^{2i}N_i - \Omega^2 A^{1i}N_i) \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(N, \Omega) = \lambda(N, AN) + M_3(\Omega^1 A^{2i}N_i - \Omega^2 A^{1i}N_i) = 0. \tag{14}$$

From (13) and (14) using $\dot{M}_3 = \lambda N_3$ we obtain

$$(N, AN)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(N, AN) + 2M_3^{-1}\frac{\mathrm{d}}{\mathrm{d}t}M_3 = \frac{\mathrm{d}}{\mathrm{d}t}\ln\{M_3^2(N, AN)\} = 0. \tag{15}$$

From (15) we get the third integral

$$J_3 = M_3^2(N, AN). \tag{16}$$

Thus equations (11) can be solved by quadratures.

We shall give a sketch of the integration in the case of homogeneous constraint, $P = 0$ and $A = \operatorname{diag}(a_1, a_2, a_3)$. Then equations (11) take the form

$$\begin{aligned}
&\dot{M}_1 = -a_2M_3M_2 + \lambda N_1 \qquad \dot{M}_2 = a_1M_3M_1 + \lambda N_2 \qquad \dot{M}_3 = \lambda N_3\\
&\dot{N}_1 = -a_2N_3M_2 \qquad \dot{N}_2 = a_1N_3M_1 \qquad \dot{N}_3 = 0\\
&a_1N_1M_1 + a_2N_2M_2 + a_3N_3M_3 = 0.
\end{aligned} \tag{11$'$}$$

The invariant surfaces

$$\begin{array}{ll} a_1N_1M_1 + a_2N_2M_2 + a_3N_3M_3 = 0 & N_3 = n \\ M_3^2(a_1N_1^2 + a_2N_2^2 + a_3N_3^2) = g & a_1M_1^2 + a_2M_2^2 + a_3M_3^2 = h \end{array} \qquad (12')$$

are diffeomorphic to a cylinder $S^1 \times R^1$.

We can introduce new variables u and v by the formulae: $u = a_1N_1^2 + a_2N_2^2 + a_3N_3^2$, $v = M_1N_2 - N_1M_2$. From equations (11′) we obtain the equations for u and v: $\dot{u} = 2a_1a_2v$, $\dot{v} = nh$, which can be easily solved. From the equation for M_3 we have $\dot{M}_3 = -a_1a_2nv(t)M_3/u(t)$. Since $u(t)$ and $v(t)$ are known functions of time, we can get $M_3(t)$. Finally, from (12′) we can find $M(t)$ and $N(t)$.

Also, we have the following generalization.

Theorem 3.2. *If the potential $U(N)$ satisfies the equation*

$$A^{2i}N_i\partial U/\partial N_1 = A^{1i}N_i\partial U/\partial N_2$$

then equations (11) are integrable. The third integral is given by (16).

An example of an integrable potential is $U(N) = \epsilon(N, AN)$.

4. The nonholonomic motion of a four-dimensional rigid body

In this section we shall find the conservation laws of the four-dimensional nonholonomic rigid body motions. The configuration space of an n-dimensional rigid body fixed at a point is $SO(n)$: $g \in SO(n)$ maps the frame attached to the body $\{e_1 = (1, 0, \ldots, 0)^t, \ldots, e_n = (0, \ldots, 0, 1)^t\}$ to the fixed frame $\{v_1 = (v_{11}, \ldots, v_{1n})^t, \ldots, v_n = (v_{n1}, \ldots, v_{nn})^t\}$ $(g = (v_1, \ldots, v_n)^t$ and $v_i = g^te_i)$. The matrix $\Omega^c = \Omega = g^{-1}\dot{g}$ is the angular velocity of the body relative to the moving frame, and $\Omega^s = \dot{g}g^{-1} = g\Omega g^{-1}$ is the angular velocity of the body in the fixed frame.

The orthonormal base with respect to the Killing form of the Lie algebra $so(n)$

$$\langle\Omega_1, \Omega_2\rangle = -\tfrac{1}{2}\,\mathrm{tr}(\Omega_1\Omega_2)$$

consists of the matrices: $\{e_i\wedge e_j,\ 1 \leqslant i < j \leqslant n\}$ where $x\wedge y = x\otimes y - y\otimes x = xy^t - yx^t$. The Killing form allows the identification $so(n)^* = so(n)$. With this identification the operation ad^* becomes the commutator of the algebra, taken with a minus sign.

Following [7], we shall consider the four-dimensional generalizations of the Veselov–Veselova problem (1). Instead of the rotations about an axis in the three-dimensional case, for the multidimensional rigid body motion we can consider rotations in the two-dimensional planes. Let us note that we can see the matrices $v_i \wedge v_j \in so(n)$ as oriented two-dimensional planes in R^n.

In (1) the projection of the angular velocity $\vec{\Omega}$ to the vector in the space $\vec{N}$ is constant. This implies that infinitesimal rotations of the body in the plane $\vec{N}^{\perp}$ are constant. By analogy, let the infinitesimal rotations of the body in the planes $v_i \wedge v_j$ be fixed, where $v_i \wedge v_j$ $(i < j)$ belong to the k-dimensional subspace $V \subset so(4)$

$$V = \mathrm{Span}\{v_i \wedge v_j,\ (i, j) \in \mathcal{V}\} \qquad \mathcal{V} \subset \{1, 2, 3, 4\} \times \{1, 2, 3, 4\} \qquad |\mathcal{V}| = k.$$

The corresponding right-invariant nonholonomic constraints are

$$\Omega^s_{ij} = \langle e_i \wedge e_j, \Omega^s\rangle = \langle e_i \wedge e_j, g\Omega g^{-1}\rangle = \langle v_i \wedge v_j, \Omega\rangle = q_{ij} \qquad (i, j) \in \mathcal{V} \qquad (17)$$

where we used $v_i \wedge v_j = g^te_i(g^te_j)^t - g^te_j(g^te_i)^t = g^{-1}e_i \wedge e_jg$. The equations of motion (2′), after the identification $so(4)^* = so(4)$, take the form:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}M = [M, \Omega] + \sum \lambda_{ij}v_i \wedge v_j \\ &\frac{\mathrm{d}}{\mathrm{d}t}(v_i \wedge v_j) = [v_i \wedge v_j, \Omega] \\ &\langle v_i \wedge v_j, \Omega\rangle = q_{ij} \qquad (i, j) \in \mathcal{V} \end{aligned} \qquad (18)$$

where $M = \mathcal{I}\Omega$ is the angular momentum of the body in the moving frame. The inertia operator $\mathcal{I}$ has the form $\mathcal{I}\Omega = J\Omega + \Omega J$ where $J = \mathrm{diag}(J_1, J_2, J_3, J_4)$ is the mass tensor [7, 12].

In the case of a dynamically symmetric rigid body, the integrability follows straightforwardly. From now on we shall suppose that $\mathcal{I} \neq c\,Id_{so(4)}$ ($c = \mathrm{const}$).

There are two functionally independent invariants on $so(4)$: $I_1(M) = \langle M, M\rangle$, $I_2(M) = \langle R(M), M\rangle$, where the operator $R : e_i \wedge e_j \to \sum r_{ij,i'j'} e_{i'} \wedge e_{j'}$ is defined by $r_{12,34} = r_{34,12} = r_{14,23} = r_{23,14} = 1$, $r_{13,24} = r_{24,13} = -1$; the rest of $r_{ij,i'j'}$ are equal to zero. In a similar way as in section 2, we can prove the following lemma.

Lemma 4.1. *Let $\Phi(M, M)$ be an invariant on $\mathcal{G}^*$ (Φ is symmetric 2-tensor) and let $H(M) = \frac{1}{2}(M - P, \Omega)$ be the Hamiltonian, then for the equations (2′) we have the following relations:*

(i) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(N^i, N^j) = 0$

(ii) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(M, M) = 2\sum_j \lambda_j \Phi(M, N^j)$

(iii) $\frac{\mathrm{d}}{\mathrm{d}t}H(M) = \sum_j \lambda_j (N^j, \Omega) = \sum \lambda_j q^j$

(iv) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(M, N^i) = \sum_j \lambda_j \Phi(N^j, N^i)$.

From lemma 4.1 (i) we get the geometric integrals

$$X_{ij,i'j'} = \langle v_i \wedge v_j, v_{i'} \wedge v_{j'}\rangle \qquad Y_{ij,i'j'} = \langle R(v_i \wedge v_j), v_{i'} \wedge v_{j'}\rangle \qquad (i, j), (i', j') \in \mathcal{V}. \tag{19}$$

We can take $X_{ij,i'j'} = \delta_{ij,i'j'}$ and $Y_{ij,ij} = r_{ij,i'j'}$.

Theorem 4.1. *(i) Let $V = V_0 \oplus V_1 \oplus V_2$, $V_1 \oplus V_2 = V \cap R(V)$, $V_2 = R(V_1)$. The following functions are integrals of the four-dimensional nonholonomic rigid body motion (18):*

$$\begin{aligned}
Z_{ij} &= \langle R(M), v_i \wedge v_j\rangle \qquad v_i \wedge v_j \in V_0 \\
J_1 &= \langle M, \Omega\rangle - 2\sum_{v_i \wedge v_j \in V} \langle v_i \wedge v_j, \Omega\rangle\langle v_i \wedge v_j, M\rangle \\
J_2 &= \langle M, M\rangle - \sum_{v_i \wedge v_j \in V} (\langle M, v_i \wedge v_j\rangle)^2 \\
J_3 &= \langle R(M), M\rangle + \langle M, M\rangle - \sum_{v_i \wedge v_j \in V_0 \oplus V_1} (\langle R(M), v_i \wedge v_j\rangle + \langle M, v_i \wedge v_j\rangle)^2.
\end{aligned} \tag{20}$$

(ii) In the $(6k + 6)$-dimensional space $so(4)\{M\} \times_{(i,j)\in\mathcal{V}} so(4)\{v_i \wedge v_j\}$ the invariant surface given by the constraints (17) and integrals (19), (20):

$$X_{ij,i'j'} = \delta_{ij,i'j'} \qquad Y_{ij,ij} = r_{ij,i'j'} \qquad Z_{ij} = z_{ij} \qquad J_i = j_i \tag{21}$$

is five-dimensional.

Sketch of proof. The first part of the theorem follows from lemma 4.1 (ii)–(iv). For the second part, a careful analyses should be done depending on $k = \dim V$ and $\dim V_0$.

Remark 4.1. The integrability of system (18) is an open problem. However, there are integrable subsystems, similar to the three-dimensional problem (1).

Lemma 4.2. *If the constraints (17) are homogeneous and for some $(i_0, j_0) \in \mathcal{V}$ the initial conditions satisfy*

$$\begin{aligned}
&M_{kl} = (v_{i_0} \wedge v_{j_0})_{kl} = 0 \qquad (k, l) = (1, 4), (2, 4), (3, 4) \\
&(v_i \wedge v_j)_{kl} = 0 \qquad (k, l) = (1, 2), (1, 3), (2, 3) \qquad (i, j) \in \mathcal{V} - \{(i_0, j_0)\}.
\end{aligned} \tag{22}$$

Then, the motion of the four-dimensional nonholonomic rigid body problem (18) is unevenly winding over the invariant tori and could be integrated by quadratures.

Proof. It could be proved that the submanifold given by (21), (22) is the two-dimensional invariant surface of equations (18). Since the connected component of the invariant surface is compact, and since for $J_1 \neq 0$ equations (18) do not have a singularity on the invariant surface, the lemma follows from the theorem about integration of the nonholonomic system with an invariant measure (see [1]). □

Maybe, one of the possible ways of proving the complete integrability is the construction of the L–A pair, as in the free rigid body motion [10]. The following crucial remark is due to Kozlov and Fedorov [7]. Considering the n-dimensional case, they wrote equations (18) with $\mathcal{V} = \mathcal{V}_{KF} = \{(i, j),\ 2 \leqslant i < j \leqslant n\}$, in an equivalent commutative form:

$$\begin{aligned} &\dot{Q} = [Q, \Omega] \qquad \dot{\Gamma} = [\Gamma, \Omega] \qquad \Gamma = v_1 \otimes v_1 \\ &Q = M_{V^\perp} + \Omega_V = (M\Gamma + \Gamma M) + \Omega - (\Omega\Gamma + \Gamma\Omega). \end{aligned} \tag{23}$$

Equations (23) have the L–A pair $\dot{L} = [L, A]$, $L = hQ + \Gamma$, $A = \Omega$, which gives the integrability only for $n = 3$. Fedorov and Kozlov suggested the hypothesis that the problem is integrable for arbitrary n.

Remark 4.2. We have the same number of integrals after the addition of a gyroscope to the rigid body (then $M = \mathcal{I}\Omega + P$). Also, as we did not use the relation $\mathcal{I}\Omega = J\Omega + \Omega J$, all the statements are valid for all operators $\mathcal{I}$: $so(n) \to so(n)$.

By the use of lemma 4.1, we can obtain analogous conservation laws for the other LR systems (2) on six-dimensional algebras $\mathcal{G}$ with quadratic invariants which belong to the two classes of algebras $\mathcal{A}$ and $\mathcal{B}$, whose description can be found in [2]. The Lie algebras $so(4)$, $so(3.1)$, $so(2.2)$, $e(3)$, $sl(2.R) \oplus sl(2.R)$ etc, are in those classes.

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References

[1] Arnol'd V I, Kozlov V V and Neishtadt A I 1987 Mathematical aspects of classical and celestial mechanics *Dynamical Systems* vol III (Berlin: Springer)

[2] Bogoyavlenski O I 1984 Integrable Euler's equations on Lie algebras, connected with problems in mathematical physics *Izv. Acad. Nauk SSSR Ser. Math.* **48** 883–938 (in Russian)

[3] Bloch A M, Krishnaprasad J E, Marsden J E and Murray R M 1996 Nonholonomic mechanical systems with symmetry *Arch. Ration. Mech. Anal.* **136** 21–99

[4] Chaplygin S A 1976 *On Rolling of a Ball on a Horizontal Plane. Collected Papers* (Moscow: Nauka) pp 409–28 (in Russian)

[5] Dragović V, Gajić B and Jovanović B 1998 Generalizations of classical integrable nonholonomic rigid body systems *J. Phys. A: Math. Gen.* **31** 9861–9

[6] Fedorov Yu N 1989 Two integrable nonholonomic systems in classical dynamics *Vestn. Mosk. Univ.* I **105** 38–41 (in Russian) 1989 *Moskow Univ. Math. Bull.* (Engl. transl.)

[7] Fedorov Yu N and Kozlov V V 1995 Various aspects of n-dimensional rigid body dynamics *Trans. Am. Math. Soc. Ser. 2* **168** 141–71

[8] Jovanović B 1998 Non-holonomic geodesic flows on Lie groups and the integrable Suslov problem on $SO(4)$ *J. Phys. A: Math. Gen.* **31** 1415–22

[9] Kozlov V V 1988 Invariant measures of the Euler–Poincaré equations on Lie algebras *Funkt. Anal. Prilozh.* **22** 69–70 (in Russian) 1988 *Funct. Anal. Appl.* **22** (Engl. transl.)

[10] Manakov S V 1976 Remarks on the integrals of the Euler equations of the n-dimensional heavy top *Funkt. Anal. Prilozh.* **10** 93–94 (in Russian) *Funct. Anal. Appl.* **10** (Engl. transl.)

[11] Neimark Yu I and Fufaev N A 1967 Dynamic of the nonholonomic systems (Moscow: Nauka) (in Russian) 1972 Translations of mathematical monographs *Am. Math. Soc.* **33** (Engl. transl.)

[12] Ratiu T 1980 The motion of the free n-dimensional rigid body *Indiana Univ. Math. J.* **29** 609–29

[13] Ratiu T 1982 Euler–Poisson equations on Lie algebras and the n-dimensional heavy rigid body *Am. J. Math.* **104** 409–48

[14] Vershik A M and Gershkovich V Ya 1994 Nonholonomic dynamical systems, geometry of distributions and variational problems *Dynamical Systems* vol VII, ed V Arnol'd and S P Novikov (Berlin: Springer)

[15] Veselov A P and Veselova L E 1986 Flows on Lie groups with nonholonomic constraint and integrable nonHamiltonian systems *Funkt. Anal. Prilozh.* **20** 65–6 (in Russian) 1986 *Funct. Anal. Appl.* **20** (Engl. transl.)

[16] Veselov A P and Veselova L E 1988 Integrable nonholonomic systems on Lie groups *Mat. Zametki* **44** 604–19 (in Russian) 1988 *Math. Notes* **44** (Engl. transl.)